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NEMO ignorat Newtonianam formulam, qua *Polynomium* quodcumque, ope binomii assumpti, ad quamvis potestatem extollitur; sed nemo, quod sciam, eam demonstravit. Hoc ego facere conatus meditatiunculas meas tibi æquissimo & optimo Judici mitto. *Tu, corrige, sodes, hoc dic, hocque, parum claris lucem dare coge, arguito ambigue dictum, mutanda notato.*

Continet hoc Problema tria prorsus diversa, quæ cum diversimode gignantur, & cum optima demonstratio e rei natura, vel genesi ducatur, diversa quoque probatione sunt confirmanda: Siquidem index est aut *integer*, aut *fractus*, uterque demum vel *positivus*, vel *negativus*.

1. Index sit *integer*, & *positivus*, tunc binomium ad potestatem cujus index est m elevare, nihil aliud est, quam toties binomium datum scribere, quoties unitas est in m, & omnia hæc binomia invicem ducere.

2. Si index est *fractus*, & *positivus*, binomium elevare ad potestatem $\frac{r}{n}$ est, datum binomium elevare ad potestatem r, &, hac potestate data, quærere quantitatem, quæ data ad potestatem n æquat ipsam dati binomii potestatem r.

N 3. Cum

3. Cum vero Index eſt *negativus*, ſive is *integer*, ſive *fractus*, ut binomium elevetur, facienda ſunt, quæ ſupra N°. 1. vel 2, & deinde per inventam poteſtatem unitas eſt dividenda.

Sumo Binomium $p+q$, ut indicet mihi quodvis Polynomium.

Inter p^m, & q^m tot ſunt medii Geometrici, in ratione $p.q$ quot unitates in $m-1$.

Hos terminos inventurus noto, quod p^m eſt ad q^m in ratione compoſita ipſius $p^m.\ 1$, & $1.q^m$, ut & p ad q habet rationem compoſitam ex $p.\ 1$, & ex $1.q$; ſed ſi fiant duæ ſeries poteſtatum, in quarum altera indices ipſius p decreſcant eadem proportione arithmetica, cujus differentia eſt 1, qua creſcunt in ſecunda ſerie indices ipſius q, habebitur ſeries continue proportionalium in ratione $p.\ 1$, & $1.q$.

$$\text{Sic } p.1 :: p^m.p^{m-1}.p^{m-2}.p^{m-3}.p^{m-4}\ldots\ldots p^{m-m}=p^0=1$$
$$1.q :: 1.q.\ \ q^2.\ q^3.\ q^4.\ \ldots\ldots\ldots\ q^m.$$

Ergo terminis reſpondentibus invicem ductis

$$p.q :: p^m.p^{m-1}q.p^{m-2}q^2.p^{m-3}q^3.p^{m-4}q^4\ldots q^m$$

Nunc dico $\overline{p+q}|^m$ componi ex terminis ſupra inventis, ut facile ex geneſi probatur.

Ergo omnes termini, qui ſunt in $\overline{p+q}|^m$ ordine diſpoſiti ſunt in proportione continua.

Et quidem duo quivis ſeſe immediate ſequentes ſunt, ut primus binomialis radicis terminus ad ſecundum.

Quod patet ex geneſi, nam p aliquoties ductum eſt ad q toties ductum in p, ut $p.q$.

Igitur omnium numerus eſt $m+1$; ſed & in ſerie arithmetica decreſcente $m.m-1.m-2.\ldots\ldots 0-$ termini ſunt numero $m+1$, aut creſcente $0.1.2.3.\ldots\ldots m$;

. . . . . m; ergo termini componentes $\overline{p+q}|^m$ debent habere indices hos, aut esse $p^m.p^{m-1}q$. . . . . . q^m.

Atqui ex legibus multiplicationis numerus terminorum debet esse $2^m > m+1$, ergo in hoc facto aliqui termini repetiti debent inveniri.

Vulgaria facta (ea, nempe, quorum multiplicans & multiplicandum constat quantitatibus diversis) omnes continent diversos terminos, quia omnes formantur diversis factoribus. In potestatibus ergo dispiciendum quinam termini diversi essent, nisi factores semper essent iidem, & quot ex diversis restitutione literarum æquales fiant; sic enim reperiemus quoties quisque in potestate repeti debeat.

Jam patet, quod si factores semper essent diversi, diversi quoque essent omnes termini in producto.

Quod cum primus in producto non fiat nisi ex primis multiplicantium, & ultimus illius ex horum ultimis, semper hæc facta erunt diversa, quamvis binomia facientia sint eadem, quia primus binomii terminus differt a secundo.

Quod ex cæteris aliqui possunt fieri æquales, quia conflantur ex primis facientium ductis in secundos, & diversimode junctis.

Igitur quærendum est, quot diversis modis jungi possint quantitates, quarum numerus datus est.

In casu nostro index rerum est m, res diversæ duæ, quarum una repetitur vicibus s, altera t, ita ut $s+t=m$; ergo numerus permutationum erit

$$\frac{m.m-1.m-2.m-3 \; . \; . \; . \; . \; . \; . \; . \; . \; 1}{s.s-1.s-2 \; . \; . \; . \; . \; 1.t.t-1.t-2.t-3 \; . \; . \; . \; 1}$$

Sic sit $t=1$, $s=m-1$, terminus erit $p^{m-1}q$, & ejus coefficiens $\dfrac{m.m-1.m-2.m-3 \; . \; . \; . \; . \; . \; 1}{m-1.m-2.m-3 \; . \; . \; . \; . \; . \; . \; 1}=m$.

 Sit

Sit $t=3, s=m-3$; habebitur coefficiens ipsius $p^{m-3}q^3, = \frac{m.m-1.m-2.m-3.m-4. \ldots \ldots 1}{1.2.3.m-3.m-4.m-5. \ldots \ldots 1} = \frac{m.m-1.m-2}{1.2.3.}$, & sic de cæteris.

Si quis forte dubitet, an superior demonstratio evincat omnes terminos necessario formari tot modis, quibus possunt, & contendat eam tantum ostendere id accidere posse, hoc responsi ferat.

Certe $\overline{p+q}|^m = \overline{p+q} \times \overline{p+q}|^{m-1}$; sed inter hujus terminos sunt $p^{m-n-1}q^n$, & $p^{m-n}q^{n-1}$, quæ necessario ducentur in p & q, & $p^{m-n-1}q^n \times p = p^{m-n}q^n = p^{m-n}q^{n-1} \times q$, ergo $p^{m-n}q^n$ omnibus modis possibilibus factum erit in $\overline{p+q}|^m$, si $p^{m-n-1}q^n$ & $p^{m-n}q^{n-1}$ sint genita quot modis possunt in $\overline{p+q}|^{m-1}$; quod necessario erit, si $p^{m-n-2}q^n$, & $p^{m-n}q^{n-2}$ sint in inferiori potestate $\overline{p+q}|^{m-2}$, & sic semper usque ad quadratum in quo pp, pq, & qq habentur, effícta tot quot possunt modis (4. II. *Euclid.*) ergo & in superioribus.

Hoc ratiocinium monet, ut idem etiam sic ostendam, ratione paulo diversa.

Jam primi coefficientem esse unitatem demonstravimus.

Secundus terminus $p^{m-1}q$ conficitur ex $p^{m-2}q \times p$, & $p^{m-1} \times q$, id est, ex primo radicis in secundum ipsius $\overline{p+q}|^{m-1}$, & ex secundo radicis in primum $\overline{p+q}|^{m-1}$, ergo in $\overline{p+q}|^m$ adest $p^{m-1}q$ semel, plus toties, quoties secundus est in $\overline{p+q}|^{m-1}$, qui ibi est semel, plus toties, quoties secundus in $\overline{p+q}|^{m-2}$, qui rursus ibi est semel plus

plus toties, quoties ſecundus eſt in $\overline{p+q}|^{m-2}$, & ſic ſemper donec deveniatur ad $\overline{p+q}|^{m-m}$, ubi ſemel eſt ſecundus; ergo quærenda eſt ſumma tot unitatum, quot ſunt in m, quæ eſt m.

Item tertius $p^{m-2}qq$ conficitur ex $p^{m-3}qq \times p$, tertio $\overline{p+q}|^{m-1}$ in primum radicis, & ex $p^{m-2}q \times q$ ſecundo ipſius $\overline{p+q}|^{m-1}$ in ſecundum radicis; ergo $\overline{p+q}|^{m}$ continebit $p^{m-2}qq$ quoties ſecundus continetur in $\overline{p+q}|^{m-1}$, id eſt, $m-1$ vices, plus toties quoties ibidem aſtat tertius, id eſt, quoties ſecundus eſt in $\overline{p+q}|^{m-2}$ $(m-2)$ plus quoties ibi eſt tertius, qui rurſus eſt quoties ſecundus eſt in $\overline{p+q}|^{m-3}$ $(m-3)$ plus quoties ibi eſt tertius, atque ita porro donec perveniamus ad $\overline{p+q}|^{2}$ ubi ſemel eſt tertius, aut ad $p+q$, ubi tertius nullus eſt; nam ſemper quærenda eſt ſumma progreſſionis arithmeticæ $m-1.m-2.m-3 \ldots\ldots 1$, aut $m-1.m-2 \ldots\ldots\ldots 0$, in illa numerus terminorum eſt $m-1$, in hac m, ut patet; quare hæc ſumma $= \overline{m-1+1} \times \frac{m-1}{2} = m \times \frac{m-1}{2} = \overline{m-1+0} \times \frac{m}{2}$.

Eodem pacto coefficientes reliquorum terminorum probabuntur efficere ſeriem in qua ſecundæ differentiæ ſunt in progreſſione arithmetica, *&c.*

Unde ſemper, ubi m eſt integer, & poſitivus, formula erit $p^m + mp^{m-1}q + \frac{m.\overline{m-1}}{2}p^{m-2}qq +$

$$\frac{m.\overline{m-1}.\overline{m-2}}{2.3}p^{m-3}q^3 + \frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{2.3.4}p^{m-4}q^4$$

$$+ \frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}.\overline{m-4}}{2.3.4.5.}p^{m-5}q^5, \ \&c.$$

Si

Si fiat $p+q=p\times\overline{1+\frac{q}{p}}$, hinc orietur ipsissima Newtoni formula; nam $\overline{p+q}|^{m}=p^{m}\times\overline{1+\frac{q}{p}}|^{m}=$

$\overline{p^{m}\times 1+\frac{m}{1}\times\frac{p}{q}+\frac{m.\overline{m-1}.}{1.2}\times\frac{p^{2}}{q^{2}}+\frac{m.m-1.m-2}{1.2.3.}\times\frac{p^{3}}{q^{3}},\&c.}=$ (si $A,B,C,D,$ &c. ponantur æquare primum, secundum, tertium, quartum, &c. cum suis quemque coefficientibus) $p^{m}\times 1+mA\frac{q}{p}+\frac{m-1}{2}B\frac{q}{p}+\frac{m-2}{3}C\frac{q}{p}$ $+\frac{m-3}{4}D\frac{q}{p}+\frac{m-4}{5}E\frac{q}{p}+\frac{m-5}{6}F\frac{q}{p}$, &c.

Quæramus nunc formulam elevandi ejusdem binomii ad potestatem $\frac{r}{n}$, ubi r & n sunt numeri integri, & ambo vel positivi, vel negativi.

Jam $p.q::p^{\frac{r}{n}}.x=\frac{p^{\frac{r}{n}}q}{p}=p^{\frac{r}{n}-1}q$, quare termini erunt $p^{\frac{r}{n}}.p^{\frac{r}{n}-1}q.p^{\frac{r}{n}-2}qq.p^{\frac{r}{n}-3}q^{3}$, &c.

Coefficientes inveniendi sint $A,B,C,D,E,$ ita ut tota $\overline{p+q}|^{\frac{r}{n}}$ radix $=Ap^{\frac{r}{n}}+Bp^{\frac{r}{n}-1}q+Cp^{\frac{r}{n}-2}qq+$ $Dp^{\frac{r}{n}-3}q^{3}+Ep^{\frac{r}{n}-4}q^{4}$, &c. ergo $\overline{p+q}|^{r}$ ($p^{r}+rp^{r-1}q+$ $\frac{r.r-1}{2}p^{r-2}qq+\frac{r.r-1.r-2}{2.3}p^{r-3}q^{3}$, &c.) $=\overline{Ap^{\frac{r}{n}}+Bp^{\frac{r}{n}-1}q+Cp^{\frac{r}{n}-2}qq,\&c.}|^{n}=A^{n}p^{r}+nA^{n-1}Bp^{r-1}q+$ $nA^{n-1}Cp^{r-2}qq+nA^{n-1}Dp^{r-3}q^{3}+nA^{n-1}Ep^{r-4}q^{4}$ &c.

$+n.n-$

$+\frac{n.\overline{n-1}}{2}A^{n-2}B^2p^{r-2}qq+n.\overline{n-1}A^{n-2}BCp^{r-3}q^3+$

$n.\overline{n-1}A^{n-2}BDp^{r-4}q^4$ &c. $+\frac{n.\overline{n-1}.\overline{n-2}}{2.3}A^{n-3}B^3p^{r-3}q^3$

$+\frac{n.\overline{n-1}.\overline{n-2}}{2}A^{n-3}B^2Cp^{r-4}q^4$ &c. $+$

$\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4}A^{n-4}B^4p^{r-4}q^4$. Atque ideo collatis terminis $1=A^n=A^{n-1}=A^{n-2}$ &c. $nB=r$, & $B=\frac{r}{n}$, $nC+\frac{n.\overline{n-2}}{2}\times\frac{rr}{nn}=\frac{r.\overline{r-1}}{2}$, & $C=\frac{r.\overline{r-n}}{2.nn}$, $nD+$ $n.\overline{n-1}\times\frac{r}{n}\times\frac{r.\overline{r-n}}{2nn}+\frac{n.\overline{n-1}.\overline{n-2}}{2.3}\times\frac{r^3}{n^3}=\frac{r.\overline{r-1}.\overline{r-2}}{2.3}$, & $D=\frac{r.\overline{r-n}.\overline{r-2n}}{2.3.n^3}$ &c.

Si ergo faciamus $\frac{r}{n}=m$, & primum terminum A, &c. revivet prior formula, & $\overline{p+q}|^{\frac{r}{n}}=\overline{p+q}|^m=p^m\times$ $1+mA\frac{q}{p}+\frac{m-1}{2}B\frac{p}{q}+\frac{m-2}{3}C\frac{q}{p}$ &c.

Extollendum sit binomium $p+q$ ad negativam potestatem, seu perfectam, seu imperfectam $-s$.

Jam $\overline{p+q}|^{-s}=\frac{1}{\overline{p+q}|^s}=\frac{1}{p^s+sp^{s-1}q+\frac{s.\overline{s-1}}{2}p^{s-2}qq \text{ \&c.}}$

$=$ (per divisionem) $\frac{1}{p}-\frac{sp^{s-1}q}{p^{2s}}-\frac{s.\overline{s-1}}{2}\times\frac{p^{s-2}qq}{p^{2s}}$

$-s.s-$

$$-\frac{s.s-1.s-2}{2.3}\times\frac{p^{s-3}q^{3}}{p^{2s}}-\frac{s.s-1.s-2.s-3}{2.3.4}\times\frac{p^{s-4}q^{4}}{p^{2s}}=$$

$$p^{-s}-sp^{-s-1}q-\frac{s.s-1}{2}p^{-s-2}qq.$$

Ex hac formula facile, ſuperiorum veſtigiis inſiſtendo, eruitur ſolemnis & generaliſſima $p^{m}\times 1+mA\frac{q}{p}+\frac{m-1}{2}B\frac{q}{p}$ &c.

Non injucundum puto, quod in hac formula, ſi $m=-2$, coefficientes erunt numeri naturales, ſi $m=-3$, trigonales, pyramidales, ſi $m=-4$ &c.

Cæterum conſtat hanc formulam ſemper dare ſeriem infinitam; ſiquidem (ſi m exponit numerum poſitivum) ultimus terminus eſſe deberet q^{-m}; ſed $p.q::p^{-m}.p^{-m-1}::p^{-m-1}q.p^{-m-2}qq$, &c. ergo ratio ipſius $p^{-m}.q^{-m}$ componi deberet ex aliquibus rationibus $p.q$, quod fieri nequit, quia $p^{-m}.q^{-m}::\frac{1}{p^{m}}.\frac{1}{q^{m}}::q^{m}.p^{m}$ in ratione compoſita ex reciprocis ipſius $p.q$.

Quod & aliter demonſtratur, indices ipſius p faciunt progreſſionem arithmeticam, cujus termini $-m$, $-m-1$, $-m-2$, &c. negativi quidem ſunt, ſed creſcunt, aut ab 3 recedunt; atqui ultimus terminus debet eſſe $q^{-m}=p^{0}q^{-m}$, ergo nunquam ad illud devenietur.

Viviaci,
poſtridie Id. *Septemb.*
CIↃ IↃCCXXXXI.

II. *Two*